Solution 4

1. A quick proof of Hölder's Inequality consists of two steps: First, assuming $\|f\|_p = \|g\|_p = 1$ and integrate Young's Inequality. Next, observe that $f/\|f\|_p$ satisfies the first step. Can you find any disadvantage of this approach?

   Solution. Following the hint, first assume $\|f\|_p = \|g\|_p = 1$. We apply Young's Inequality to get
$$|f(x)g(x)| \le \frac{|f(x)|^p}{p} + \frac{|g(x)|^q}{q} \ .$$
   Then integrate, using $\|f\|_p = \|g\|_p = 1$ and $1/p + 1/q = 1$ to get the Hölder's Inequality in the form
$$\int_a^b |fg|\, dx \le 1.$$
   Next, since $F = f/\|f\|_p$ and $G = g/\|g\|_q$ satisfy the conditions in the first step. We have
$$\int_a^b |FG|\, dx \le 1 \ .$$
   Writing back in f and g, we get the desired Hölder's Inequality.

   Note. A disadvantage of this proof, in my opinion, is that it cannot yield the characterization of the case of equality.

2. Prove the generalized Hölder Inequality: For $f_1, f_2, \cdots, f_n \in R[a,b]$,
$$\int_a^b |f_1 f_2 \cdots f_n| dx \le \left(\int_a^b |f_1|^{p_1}\right)^{1/p_1} \left(\int_a^b |f_2|^{p_2}\right)^{1/p_2} \cdots \left(\int_a^b |f_n|^{p_n}\right)^{1/p_n} ,$$
   where
$$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_n} = 1, \quad p_1, p_2, \cdots, p_n > 1 \ .$$

   Solution. Induction on n. $n = 2$ is the original Hölder, so it holds. Let
$$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_{n+1}} = 1 \ .$$
   First, using the original Hölder, we have
$$\int_a^b |f_1 f_2 \cdots f_{n+1}|\, dx \le \left(\int_a^b |f_1|^{p_1}\, dx\right)^{1/p_1} \left(\int_a^b |f_2 \cdots f_{n+1}|^q\, dx\right)^{1/q} ,$$
   where q is conjugate to p_1. It is easy to see
$$1 = \frac{q}{p_2} + \cdots + \frac{q}{p_{n+1}} \ .$$
   By induction hypothesis,
$$\int_a^b |f_2^q \cdots f_n^q|\, dx \le \left(\int_a^b |f_2|^{p_2}\, dx\right)^{1/p_2} \cdots \left(\int_a^b |f_{n+1}|^{p_{n+1}}\, dx\right)^{1/p_{n+1}} ,$$
   done.

3. Establish the inequality, for $f \in R[a,b]$,

$$\int_a^b |f|\,dx \le (b-a)^{1/q} \int_a^b |f|^p\,dx, \quad 1/p + 1/q = 1, p > 1 \;.$$

Solution. This is a special case of the next problem, taking $p_1 = 0$ and $p = 1$.

4. For $p \in [p_1, p_2], p_1 \ge 0$, establish the inequality, for $f \in R[a,b]$,

$$\int_a^b |f|^p\,dx \le \left(\int_a^b |f|^{p_1}\right)^{\lambda} \left(\int_a^b |f|^{p_2}\right)^{1-\lambda}, \quad p = \lambda p_1 + (1-\lambda)p_2 \;.$$

Solution. The cases $p = p_1$ or $p = p_2$ are trivial. Now, observe that $p \in (p_1, p_2)$ can be written as $p = \lambda p_1 + (1-\lambda)p + 2$ for some $\lambda \in (0,1)$. We have

$$\begin{aligned} \int_a^b |f|^p\,dx &= \int_a^b |f|^{\lambda p_1 + (1-\lambda)p_2}\,dx \\ &\le \left(\int_a^n |f|^{\lambda p_1 \lambda^{-1}}\,dx\right)^{\lambda} \left(\int_a^n |f|^{(1-\lambda)p_2(1-\lambda)^{-1}}\,dx\right)^{1-\lambda} \\ &= \left(\int_a^n |f|^{p_1}\,dx\right)^{\lambda} \left(\int_a^n |f|^{p_2}\,dx\right)^{1-\lambda} \end{aligned}$$

5. In a metric space (X,d), its metric ball is the set $\{y \in X : \; d(y,x) < r\}$ where x is the center and r the radius of the ball. May denote it by $B_r(x)$. Draw the unit metric balls centered at the origin with respect to the metrics d_2, d_∞ and d_1 on $\mathbb{R}^2$.

   Solution. The unit ball $B_1^2(0)$ is the standard one, the unit ball in d_∞-metric consists of points (x,y) either $|x|$ or $|y|$ is equal to 1 and $|x|, |y| \le 1$, so $B_1^\infty(0)$ is the unit square. The unit ball $B_1^1(0)$ consists of points (x,y) satisfying $|x|+|y| \le 1$, so the boundary is described by the curves $x+y = 1, x,y \ge 0,\; x-y = 1, x \ge 0, y \le 0, -x+y = 1, x \le 0, y \ge 0$, and $-x-y = 1, x,y \le 0$. The result is the tilted square with vertices at $(1,0), (0,1), (-1,0)$ and $(0,-1)$.

6. Show that $\|a\| = \left(\sum_j |a_j|^p\right)^{1/p}$ is no longer a norm for $p \in (0,1)$ in $\mathbb{R}^n$.

   Solution. Although the first two axioms of a norm hold but the last one is bad. For example, take $a = (1,0), b = (0,1)$ in $\mathbb{R}^2$. We have $|a|_p = |b|_p = 1$ so $|a|_p + |b|_p = 2$ but $|a+b|_p = |(1,1)|_p = 2^{1/p} > |a|_p + |b|_p$, the inequality is reversed !

7. Determine the metric ball of radius r in (X,d) where d is the discrete metric, that is, $d(x,y) = 1$ if $x \ne y$.

   Solution. When $r \in (0,1]$, $B_r(x) = \{x\}$. When $r > 1, B_r(x) = X$.

8. Let l^p consist of all sequences $\{a_n\}$ satisfying $\sum_n |a_n|^p < \infty$. Show that

$$\|a\|_p = \left(\sum_n |a_n|^p\right)^{1/p},$$

   defines a norm on $l^p, 1 \le p < \infty$. Propose a definition for the metric space l^∞.

Fix n. By the Minkowski inequality in $\mathbb{R}^n$,

$$\begin{aligned}\left(\sum_{j=1}^{n}|a_j+b_j|^p\right)^{1/p} &\leq \left(\sum_{j=1}^{n}|a_j|^p\right)^{1/p}+\left(\sum_{j=1}^{n}|b_j|^p\right)^{1/p}\\ &\leq \left(\sum_{j=1}^{\infty}|a_j|^p\right)^{1/p}+\left(\sum_{j=1}^{\infty}|b_j|^p\right)^{1/p},\end{aligned}$$

and the desired triangle inequality in l^p follows by letting $n\to\infty$.

One can define l^∞ to be the vector space consisting of all bounded sequences. It is a normed one under $\|a\| = \sup_{j\geq 1}|a_j|$.

9. Define d on $\mathbb{Z}\times\mathbb{Z}$ by $d(n,m)=2^{-d}$, where d is the largest power of 2 dividing $n-m\neq 0$ and set $d(n,n)=0$. Verify that d defines a metric on $\mathbb{Z}$.

   Solution. Noticing that the function d is positive unless $n=m$, **M1** and **M2** are clearly satisfied. If 2^d divides $m-k$ and $k-n$, then 2^d divides $m-n=m-k+k-n$. Hence

   $$d(m,n)\leq \max(d(m,k),d(k,n))\leq d(m,k)+d(k,n),$$

   and **M3** is also satisfied.

Elementary Inequalities for Functions.

We start with the Young's Inequality covered in MATH2060.

Young's Inequality. *For $a, b > 0$ and $p > 1$,*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \ , \quad \frac{1}{p} + \frac{1}{q} = 1,$$

and equality sign holds if and only if $a^p = b^q$.

The number q is called the conjugate of p. Note that $q > 1$. The proof of this inequality is left to you. Basically, we use calculus to show the function

$$\varphi(a) = \frac{a^p}{p} + \frac{b^q}{q} - ab \ ,$$

where b is fixed, has a unique minimum over $(0, \infty)$ at the point $a = b^{1/(1-p)}$, that is, $a^p = b^q$.

Hölder's Inequality. *Let $f, g \in R[a, b]$ and $p > 1$. Then*

$$\int_a^b |f(x)g(x)| \, dx \leq \left(\int_a^b |f(x)|^p \, dx \right)^{1/p} \left(\int_a^b |g(x)|^q \, dx \right)^{1/q} \ , \quad q \text{ is conjugate to } p \ .$$

Equality sign in this inequality holds if and only if either (a) f or g vanish almost everywhere, or (b) there is some positive λ such that $|g|^q = \lambda |f|^p$ almost everywhere.

Proof. Assume $\|f\|_p$ or $\|g\|_q$ is non-zero, otherwise the inequality holds trivially. In the following it is assumed $\|f\|_p > 0$. You may also assume the other case, it is symmetric anyway.

For $\varepsilon > 0$ to be chosen, by Young's Inequality,

$$|f(x)g(x)| = |\varepsilon f(x) \varepsilon^{-1} g(x)| \leq \frac{\varepsilon^p |f(x)|^p}{p} + \frac{\varepsilon^{-q} |g(x)|^q}{q} \ .$$

Integrate this inequality to get

$$\int_a^b |f(x)g(x)| \, dx \leq \frac{\varepsilon^p}{p} \int_a^b |f(x)|^p \, dx + \frac{\varepsilon^{-q}}{q} \int_a^b |g(x)|^q \, dx \ . \tag{1}$$

We now choose ε so that

$$\varepsilon^p \int_a^b |f(x)|^p \, dx = \varepsilon^{-q} \int_a^b |g(x)|^q \, dx \ ,$$

that is,

$$\varepsilon^{p+q} = \left(\int_a^b |g(x)|^q \, dx \right) \left(\int_a^b |f(x)|^p \, dx \right)^{-1} \ .$$

Using this epsilon to plug in (1), the right hand side becomes

$$\frac{\varepsilon^p}{p} \int_a^b |f(x)|^p \, dx + \frac{\varepsilon^{-q}}{q} \int_a^b |g(x)|^q \, dx \ = \left(\int_a^b |f(x)|^p \, dx \right)^{1/p} \left(\int_a^b |g(x)|^q \, dx \right)^{1/q} \ . \tag{2}$$

The Hölder's Inequality follows.

To characterize the inequality sign in this inequality, observe case (a) is obvious so let us assume $\|f\|_p, \|g\|_q$ are both positive, so $|f(x)|, |g(x)|$ are positive almost everywhere. From (1) and (2) we see that the inequality sign in (1) becomes equality, that is,

$$\int_a^b \left(\frac{\varepsilon^p |f(x)|^p}{p} + \frac{\varepsilon^{-q}|g(x)|^q}{q} - |f(x)g(x)| \right) dx = 0 \ .$$

The integrand is a non-negative function by Young's Inequality. The vanishing of this integral implies that the integrand must vanish almost everywhere, that is,

$$\frac{\varepsilon^p |f(x)|^p}{p} + \frac{\varepsilon^{-q} g(x)^q}{q} - |f(x)g(x)| = 0 \ a.e. \ .$$

By the equality sign condition in Young's Inequality, we conclude that

$$\varepsilon^p |f(x)|^p = \varepsilon^{-q} g(x)^q \ a.e.,$$

that is, $|g(x)|^q = \lambda |f(x)|^p$ almost everywhere where $\lambda = \varepsilon^{-p-q}$.

Remarks. (a) We have used the following proposition proved in Chapter 1: For $f \in R[a,b]$,

$$\int_a^b |f| \, dx = 0 \text{ if and only if } f = 0 \ a.e. \ .$$

We also point out, when $f \in C[a,b]$,

$$\int_a^b |f| \, dx = 0 \text{ if and only if } f = 0 \text{ everywhere } \ .$$

(b) When f and g in Hölder's Inequality are continuous, almost everywhere in the characterization of equality sign becomes everywhere.

(c) The inequality still holds in the limiting cases. In fact, when $g \in C[a,b]$ and $p = 1$, we have

$$\int_a^b |f(x)g(x)| \, dx \le \int_a^b |f(x)| \, dx \|g\|_\infty \ .$$

When $f \in C[a,b]$ and $p = \infty$,

$$\int_a^b |f(x)g(x)| \, dx \le \|f\|_\infty \int_a^b |g(x)| \, dx \ .$$

But there is no clean characterization of the equality sign.

Minkowski's Inequality. *For $f, g \in R[a,b]$ and $p > 1$,*

$$\|f+g\|_p \le \|f\|_p + \|g\|_p \ .$$

Equality sign in this inequality holds if and only if either (a) f or g vanishes almost everywhere, or (b) $\|f\|_p, \|g\|_p > 0$ and there is some positive λ such that $g(x) = \lambda f(x)$ almost everywhere.

Proof. Using

$$|f+g|^p = |f+g|^{p-1}|f+g| \le |f+g|^{p-1}|f| + |f+g|^{p-1}|g| \ ,$$

integrate both sides to get

$$\int_a^b |f+g|^p \, dx \le \int_a^b |f+g|^{p-1}|f| \, dx + \int_a^b |f+g|^{p-1}|g| \, dx \ . \tag{3}$$

Applying the Hölder's Inequality to the two integrals on the right separately, we have

$$\int_a^b |f+g||f| \, dx \le \left(\int_a^b |f+g|^q \, dx \right)^{1/q} \left(\int_a^b |f|^p \, dx \right)^{1/p} ,$$

and

$$\int_a^b |f+g||g| \, dx \le \left(\int_a^b |f+g|^q \, dx \right)^{1/q} \left(\int_a^b |g|^p \, dx \right)^{1/p} ,$$

where q is conjugate to p. Putting this back to (3), we obtain the desired inequality after some simplifications.

The equality case, in principle, could be treated as in the Hölder's case. It is easy to get $|g(x)|^q = \lambda |f(x)|^p$ almost everywhere, but rather tedious (or need to use Lebsegue integral) to get $f(x)^p = \lambda g(x)^q$. Luckily, this property has no consequence in our later development.